Periodic tight wavelet frames with good time-frequency localization*

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Abstract

A family of normalized tight periodic wavelet frames is constructed. The family has optimal time-frequency localization (in the sense of the Breitenberger uncertainty constant) with respect to a family parameter and it has the best currently known localization with respect to a multiresolution analysis parameter.

Keywords: periodic wavelet, scaling function, tight frame, uncertainty principle, Poisson summation formula, localization

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1 Introduction

During the last years the wavelet theory of periodic functions has been continuously refined. First, periodic wavelets were generated by periodization of wavelet functions on the real line (see, for example, [6]). A wider and more natural approach providing a flexibility on a theoretical front and in applications is to study periodic wavelets directly using a periodic analog of a multiresolution analysis and a unitary extension principle. The concept of periodic multiresolution analysis (MRA) is introduced and discussed in [12, 14, 15, 16, 21]. In [9], a unitary extension principle (UEP) for constructing normalized tight wavelet frames is rewritten for periodic functions (see Theorem 1). Later, the approach is developed in [8].

In this paper we focus on a property of good localization of both periodic wavelet functions and their Fourier coefficients. The quantitative characteristic of this property is an uncertainty constant (UC). It is introduced by Breitenberger in [3] using physical reasons as the product of frequency and angular variances (see Definition 1). The smaller uncertainty constant corresponds to the better localization. But there exists a universal lower bound for the Breitenberger uncertainty constant. This is the essence of the Breitenberger uncertainty principle. Furthermore, there is no extremal function for this principle, i.e. there is no function attending the lower bound. The proof of this result can be found in [17]. So, to find a sequence of functions having an asymptotically minimal UC and some additional setup, for example a wavelet structure, is a natural concern.

Some papers in this direction include [10, 14, 17, 20]. In [14] for a wide class of scaling and wavelet functions the following asymptotical equality is established: $UC(\varphi_n) = O(\sqrt{n})$, $UC(\psi_n) = O(\sqrt{n})$, where n is the dimension of MRA space V_n. For the first time the uniformly bounded uncertainty constants are computed in [20] for so-called trigonometric wavelets (see also [19]). In [10], it is shown that the uncertainty products of uniformly local, uniformly regular and uniformly stable scaling functions and wavelets are uniformly bounded from above by a constant. In [17] an example of an asymptotically optimal sequence of periodic functions φ_h for the periodic uncertainty principle is constructed, namely $UC(\varphi_h) < 1/2+\sqrt{h}/2$. Later, each item φ_h of the sequence is used as a scaling function to generate a stationary interpolatory MRA (V_n). For the corresponding wavelet functions $\psi_{n,h}$ the UC is optimal for a fixed space V_n, but the estimate is nonuniform with respect to n, namely $UC(\psi_{n,h}) < 1/2 + 1.1n^2\sqrt{h}$ with arbitrary $h > 0$. The situation is the same after orthogonalization: $UC(\psi^{\perp}_{n,h}) < 1/2 + 1.1n^2\sqrt{h}$, $UC(\varphi^{\perp}_{n,h}) < 1/2 + n^2\sqrt{h}$.

In this paper (Theorem 3) we construct a family of scaling sequences $\Phi^0 = \{(\varphi^a_j)_j \ : \ a > 1\}$ generating a family of wavelet sequences $\Psi^0 = \{(\psi^a_j)_j \ : \ a > 1\}$ corresponding to a nonstationary periodic MRA as it is defined in [12], [21] and [8]. Each scaling sequence has an asymptotically optimal UC with respect to the dimension of MRA spaces (see Definition 2). The UC of each wavelet sequence tends to 3/2. It is the smallest currently known value of the UC for periodic wavelet frames. The convergence is uniform with respect to the parameter a, namely

$$\lim_{j\to\infty} \sup_{a>1} UC(\varphi^a_j) = \frac{1}{2}, \quad \lim_{j\to\infty} UC(\psi^a_j) = \frac{3}{2}.$$

But the main property of the families is the optimality of the UC for all functions of a sequence $(\varphi^a_j)_j$ and simultaneously with respect to the parameter a. Moreover, it turns out that this property is inherited by the family Ψ^0. Thus, (see

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Definition 2) we get

$$\lim_{a\to\infty}\sup_{j\in\mathbb{N}} UC(\varphi_j^a) = \frac{1}{2}, \quad \lim_{a\to\infty} UC(\psi_j^a) = \frac{1}{2}.$$

This issue in principle answers the question stated in [17] whether there exists a translation invariant basis of a wavelet space W_j which is asymptotically optimal independent of the MRA level j. In other words, we are looking for a family of scaling and wavelet sequences having the asymptotic optimality for every element of the sequence. In Theorem 3 we get an affirmative answer for the scaling sequences and for tight wavelet frames with unitary bound instead of basis.

We will consider the particular issue for wavelet sequences in Section 4.

2 Notations and auxiliary results

Let $L_2(0,\,1)$ be the space of all 1-periodic square-integrable complex-valued functions, with inner product $(\cdot,\cdot)$ given by $(f,\,g) := \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$ for any $f, g, \in L_2(0,\,1)$, and norm $\|\cdot\| := \sqrt{(\cdot,\,\cdot)}$.

The Fourier series of a function $f \in L_2(0,\,1)$ is defined by $\sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}kx}$, where its Fourier coefficient is defined by $\widehat{f}(k) = \int_0^1 f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x$.

Let H be a separable Hilbert space. If there exist constants $A,\,B > 0$ such that for any $f \in H$ the following inequality holds $A\|f\|^2 \le \sum_{n=1}^\infty |(f,\,f_n)|^2 \le B\|f\|^2$, then the sequence $(f_n)_{n\in\mathbb{N}}$ is called a frame for H. If $A = B = 1$, then the sequence $(f_n)_{n\in\mathbb{N}}$ is called a normalized tight frame for H.

A frame $(f_n)_{n\in\mathbb{N}}$ is a spanning set, i.e. for any function $f \in H$ there exists a sequence $(\alpha_k)_k$ such that $f = \sum_k \alpha_k f_k$. Moreover, if in the case of a normalized tight frame it is known that $\|f_n\| = 1$ for all $n \in \mathbb{N}$, then the system forms an orthonormal basis. More information about frames can be found in [4].

In the sequel, we use the following notation $f_{j,k}(x) := f_j(x - 2^{-j}k)$ for a function $f_j \in L_2(0,\,1)$. Consider functions $\varphi_0 = 1$, $\psi_j \in L_2(0,\,1)$, $j = 0, 1, \ldots$ If the collection $\Psi := \left\{\varphi_0, \psi_{j,k} :\ j = 0,1,\ldots,\ k = 0,\ldots,2^j-1\right\}$, forms a frame (or basis) for $L_2(0,\,1)$ then Ψ is said to be a periodic wavelet frame (or wavelet basis) for $L_2(0,\,1)$.

Let us recall the UEP for a periodic setting.

Theorem 1 ([9], [15]) *Let $\varphi_j \in L_2(0,\,1)$, $j = 0,1,\ldots$, $\varphi_0 = 1$, be a sequence of 1-periodic functions such that*

$$\lim_{j\to\infty} 2^{j/2}\widehat{\varphi}_j(k) = 1. \tag{1}$$

There exists a two-parametric sequence μ_k^j such that $\mu_{k+2^j}^j = \mu_k^j$, and

$$\widehat{\varphi}_{j-1}(k) = \mu_k^j \widehat{\varphi}_j. \tag{2}$$

Let ψ_j, $j = 0,1,\ldots$, be a sequence of 1-periodic functions defined using Fourier coefficients

$$\widehat{\psi}_j(k) = \lambda_k^{j+1}\widehat{\varphi}_{j+1}(k), \tag{3}$$

where $\lambda_{k+2^j}^j = \lambda_k^j$ and

$$\begin{pmatrix} \mu_k^j & \mu_{k+2^{j-1}}^j \\ \lambda_k^j & \lambda_{k+2^{j-1}}^j \end{pmatrix} \begin{pmatrix} \overline{\mu}_k^j & \overline{\lambda}_k^j \\ \overline{\mu}_{k+2^{j-1}}^j & \overline{\lambda}_{k+2^{j-1}}^j \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{4}$$

Then the family $\Psi := \left\{\varphi_0, \psi_{j,k} :\ j = 0,1,\ldots,\ k = 0,\ldots,2^j-1\right\}$ forms a normalized tight wavelet frame for $L_2(0,\,1)$.

The sequences $(\varphi_j)_j$ and $(\psi_j)_j$ are called scaling and wavelet sequences respectively. For fixed $j = 0,1,\ldots$ the periodic sequences $(\mu_k^j)_k$ and $(\lambda_k^j)_k$ are called masks and wavelet masks respectively. (To be more precise, $(\mu_k^j)_k$ and $(\lambda_k^j)_k$ are discrete Fourier transforms for periodic analogs of masks and wavelet masks, but for the sake of brevity we use the aforementioned notations.)

This setup generates a notion of periodic MRA in the following way: By definition, put $V_j = \operatorname{span}\left\{\varphi_{j,k}; k = 0,\ldots,2^j-1\right\}$ for $j \ge 0$. Then the sequence $(V_j)_{j\ge0}$ is a periodic MRA.

Originally, the concept of an uncertainty principle was introduced for the real line case in 1927. The (Heisenberg) uncertainty constant of $f \in L_2(\mathbb{R})$ is the functional $UC_H(f) := \Delta_f\Delta_{\widehat{f}}$ such that

$$\Delta_f^2 := \|f\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}(t - t_{0f})^2|f(t)|^2\,\mathrm{d}t, \quad \Delta_{\widehat{f}}^2 := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}(\omega - \omega_{0\widehat{f}})^2|\widehat{f}(\omega)|^2\,\mathrm{d}\omega,$$

$$t_{0f} := \|f\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}} t|f(t)|^2\,\mathrm{d}t, \quad \omega_{0\widehat{f}} := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})}\int_{\mathbb{R}}\omega|\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

Let us recall the definition of the uncertainty constant and the uncertainty principle for a periodic setting.

Definition 1 ([3]) *Let $f(x)=\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{2\pi\mathrm{i}kx}$ be a 1-periodic function $f\in L_2(0,\,1)$. The first trigonometric moment is defined as*

$$\tau(f) := -2\pi \int_0^1 \mathrm{e}^{2\pi\mathrm{i}x}|f(x)|^2\,\mathrm{d}x = -2\pi\sum_{k\in\mathbb{Z}} c_k\bar{c}_{k+1}.$$

The angular variance of the function f is defined by

$$\mathrm{var_A}(f) := \frac{1}{4\pi^2}\left(\frac{\left(\sum_{k\in\mathbb{Z}}|c_k|^2\right)^2}{\left|\sum_{k\in\mathbb{Z}} c_k\bar{c}_{k+1}\right|^2}-1\right) = \frac{\|f\|^4}{|\tau(f)|^2}-\frac{1}{4\pi^2}.$$

The frequency variance of the function f is defined by

$$\mathrm{var_F}(f) := \frac{4\pi^2\sum_{k\in\mathbb{Z}}k^2|c_k|^2}{\sum_{k\in\mathbb{Z}}|c_k|^2} - \frac{4\pi^2\left(\sum_{k\in\mathbb{Z}}k|c_k|^2\right)^2}{\left(\sum_{k\in\mathbb{Z}}|c_k|^2\right)^2} = \frac{\|f'\|^2}{\|f\|^2}+\frac{(f',\,f)^2}{\|f\|^4}.$$

The quantity

$$UC(\{c_k\}) := UC(f) := \sqrt{\mathrm{var_A}(f)\mathrm{var_F}(f)}$$

is called the periodic (Breitenberger) uncertainty constant.

The following uncertainty principle holds in the periodic setting.

Theorem 2 ([3, 17]) *Let $f\in L_2(0,\,1)$ and $f(x)\neq C\mathrm{e}^{\mathrm{i}kx}$, $C\in\mathbb{R}$, $k\in\mathbb{Z}$. Then $UC(f)>1/2$ and there is no function satisfying the equality $UC(f)=1/2$.*

Since periodic wavelet bases and frames are nonstationary in nature and the uncertainty constant has no extremal function, it is natural to give the following

Definition 2 *Suppose that φ_j (ψ_j) is a scaling (a wavelet) sequence. Then the quantity*

$$\limsup_{j\to\infty} UC(\varphi_j) \quad (\limsup_{j\to\infty} UC(\psi_j))$$

is called the uncertainty constant of the scaling (the wavelet) sequence. We say that a sequence of periodic functions $(f_j)_{j\in\mathbb{N}}$ has an optimal uncertainty constant if

$$\lim_{j\to\infty} UC(f_j) = 1/2.$$

3 Main results

In the following theorem we construct a family of periodic normalized tight wavelet frames with optimal uncertainty constants for scaling functions and wavelets.

Theorem 3 *There exists a family of periodic wavelet sequences $\Psi_a := \{(\psi_j^a)_j\}_a$ such that for any fixed $a>1$ the system $\{\varphi_0^a\}\cup\{\psi_{j,k}^a:\ j=0,1,\ldots,\ k=0,\ldots,2^j-1\}$ forms a normalized tight frame in $L_2(0,\,1)$ and*

$$\lim_{j\to\infty}\sup_{a>1} UC(\varphi_j^a) = \frac{1}{2}, \quad \lim_{a\to\infty}\sup_{j\in\mathbb{N}} UC(\varphi_j^a) = \frac{1}{2}, \tag{5}$$

$$\lim_{j\to\infty} UC(\psi_j^a) = \frac{3}{2}, \quad \lim_{a\to\infty} UC(\psi_j^a) = \frac{1}{2}. \tag{6}$$

The following lemma provides a construction of the wavelet family stated in Theorem 3.

Lemma 1 *Let $\nu_k^{j,a}$ be a sequence given by*

$$\nu_k^{j,a} := \begin{cases} \exp\left(-\frac{k^2+a^2}{(j-1)(j-2)a}\right), & k=-2^{j-2}+1,\ldots,2^{j-2}, \\ \sqrt{1-\exp\left(-\frac{2((k-2^{j-1})^2+a^2)}{(j-1)(j-2)a}\right)}, & k=2^{j-2}+1,\ldots,3\times 2^{j-2}, \end{cases} \tag{7}$$

and extended 2^j-periodic with respect to k. Furthermore we define $\widehat{\xi}_j^a(k) := \prod_{r=j+1}^{\infty}\nu_k^{r,a}$. Then the scaling sequence, masks, wavelet masks and wavelet sequence are defined respectively as

$$\widehat{\varphi_j^a}(k) := 2^{-j/2}\widehat{\xi}_j^a(k), \qquad \mu_k^{j,a} := \sqrt{2}\nu_k^{j,a},$$
$$\lambda_k^{j,a} := e^{2\pi i 2^{-j}k}\mu_{k+2^{j-1}}^{j,a}, \qquad \widehat{\psi}_j^a(k) := \lambda_k^{j+1,a}\widehat{\varphi}_j^a(k) \tag{8}$$

and the family $\Psi_a := \left\{\mathbf{1},\psi_{j,k}^a:\ j=0,1,\ldots,\ k=0,\ldots,2^j-1\right\}$ satisfies the conditions of Theorem 3.

Proof. Using definition (7) and the elementary identity $(j-1)^{-1}(j-2)^{-1} = (j-2)^{-1} - (j-1)^{-1}$, we get

$$\widehat{\xi}_j^a(k) = \begin{cases} \prod_{r=j+1}^{J-1} \nu_k^{r,a} \prod_{r=J}^{\infty} \nu_k^{r,a} = \left(\prod_{r=j+1}^{J-1} \nu_k^{r,a}\right) \exp\left(-\frac{k^2+a^2}{(J-2)a}\right), & j \le J-2, \\ \prod_{r=j+1}^{\infty} e^{-\frac{k^2+a^2}{(r-2)(r-1)a}} = \exp\left(-\frac{k^2+a^2}{(j-1)a}\right), & j > J-2, \end{cases} \tag{9}$$

where $J = \lfloor \log_2 |k-1| + 3 \rfloor$. Therefore, the coefficients $\widehat{\xi}_j^a(k)$ are well-defined. Then a straightforward calculation shows that conditions (1)-(4) hold. Consequently, the sequence $(\nu_k^{j,a})_{j,k}$ defines scaling and wavelet sequences in the aforementioned way (8) and the family $\Psi_a := \{\mathbf{1}, \psi_{j,k}^a : \ j = 0,1,\dots, \ k = 0,\dots,2^j-1\}$ forms a normalized tight wavelet frame for $L_2(0,\,1)$ for any fixed $a > 1$.

Let us check equalities (5) and (6). First, the uncertainty constant is a continuous functional with respect to the norm $\|f\|_{W_1^2} := \|f\|^2 + \|f'\|^2$. Indeed, let us check that $\tau(f)$ and $(f',\,f)$ are continuous functionals with respect to this norm. It follows from elementary properties of integrals and the Cauchy-Bunyakovskiy-Schwarz inequality that

$$\begin{aligned} |\tau(f) - \tau(g)| &= 2\pi \left| \int_0^1 \mathrm{e}^{2\pi \mathrm{i} x} |f(x)|^2 \,\mathrm{d}x - \int_0^1 \mathrm{e}^{2\pi \mathrm{i} x} |g(x)|^2 \,\mathrm{d}x \right| \\ &\le 2\pi \int_0^1 \left| |f(x)|^2 - |g(x)|^2 \right| \mathrm{d}x \le 2\pi \left(\Big| |f| - |g| \Big|,\, |f| + |g| \right) \le 2\pi \Big\| |f| - |g| \Big\| \, \Big\| |f| + |g| \Big\| \\ &\le 2\pi \|f - g\| \, \Big\| |f| + |g| \Big\| \le 2\pi \Big(\|f\| + \|g\| \Big) \|f - g\|_{W_1^2} \end{aligned}$$

and

$$\begin{aligned} \big|(f',\,f) - (g',\,g)\big| &\le \big|(f',\,f) - (f',\,g) + (f',\,g) - (g',\,g)\big| \le |(f',\,f-g)| + |(f'-g',\,g)| \\ &\le \|f'\|\,\|f-g\| + \|f'-g'\|\,\|g\| \le \max\Big\{\|f'\|, \|g'\|\Big\} \|f-g\|_{W_1^2}. \end{aligned}$$

Since the uncertainty constant continuously depends on $\|f\|$, $\|f'\|$, $\tau(f)$, and $(f',\,f)$, the result can be proved by direct calculations.

The uncertainty constant is a homogeneous functional, i.e. $UC(\alpha f) = UC(f)$ for any $\alpha \in \mathbb{R}$, so $UC(\phi_j^a) = UC(2^{-j/2}\xi_j^a) = UC(\xi_j^a)$ and in the sequel we prove the equalities $\lim_{j\to\infty} \sup_{a>1} UC(\xi_j^a) = 1/2$ and $\lim_{a\to\infty} \sup_{j\in\mathbb{N}} UC(\xi_j^a) = 1/2$ instead of (5).

Let us denote

$$\xi_j^{a,0}(x) := \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\frac{k^2+a^2}{(j-1)a}} \mathrm{e}^{2\pi \mathrm{i} k x} = \mathrm{e}^{-\frac{a}{j-1}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\frac{k^2}{(j-1)a}} \mathrm{e}^{2\pi \mathrm{i} k x}.$$

Since the UC is a homogeneous functional,

$$UC(\xi_j^{a,0}) = UC\left(\left\{\mathrm{e}^{-\frac{k^2}{(j-1)a}}\right\}\right).$$

It is known (see [17]) that

$$\lim_{j\to\infty} UC\left(\left\{\mathrm{e}^{-\frac{k^2}{j}}\right\}\right) = 1/2.$$

Substituting $(j-1)a$ for j it is clear that for any $a > 1$ the equality is preserved and the rate of convergence does not decay as $a \to \infty$. Thus, it is fulfilled that

$$\lim_{j\to\infty} \sup_{a>1} UC(\xi_j^{a,0}) = \frac{1}{2}, \qquad \lim_{a\to\infty} \sup_{j\in\mathbb{N}} UC(\xi_j^{a,0}) = \frac{1}{2}.$$

So, taking into account the continuity of the uncertainty constant, it is sufficient to check that $\lim_{j\to\infty} \|\xi_j^a - \xi_j^{a,0}\|_{W_1^2} = 0$ uniformly on $a > 1$, and $\lim_{a\to\infty} \|\xi_j^a - \xi_j^{a,0}\|_{W_1^2} = 0$ uniformly on $j \in \mathbb{N}$.

Using the definition of $\nu_k^{r,a}$, we get

$$\widehat{\xi}_j^a(k) = \prod_{r=j+1}^{\infty} \nu_k^{r,a} = \prod_{r=j+1}^{\infty} \exp\left(-\frac{k^2+a^2}{(r-1)(r-2)a}\right) = \exp\left(-\frac{k^2+a^2}{(j-1)a}\right) = \widehat{\xi}_j^{a,0}(k) \tag{10}$$

for $k = -2^{j-2}+1, \dots, 2^{j-2}$. Applying the elementary property of series to the Fourier coefficients $\widehat{\xi}_j^a(k)$ and $\widehat{\xi}_j^{a,0}(k)$, (namely, if $\lim_{j\to\infty}(a_j(0) - b_j(0)) = 0$ and $\lim_{j\to\infty} \sum_{k\in\mathbb{Z}} k^2\,|a_j(k) - b_j(k)|^2 = 0$, then $\lim_{j\to\infty} \sum_{k\in\mathbb{Z}} |a_j(k) - b_j(k)|^2 = 0$) we see that it is sufficient to check the following equality $\lim_{j\to\infty} \|(\xi_j^a)' - (\xi_j^{a,0})'\| = 0$.

Let us consider coefficients $\widehat{\xi}_j^a(k)$ and $\widehat{\xi}_j^{a,0}(k)$ for $k \ge 2^{j-2}+1$, that is for $j \ge J-1$. Using (9), we obtain

$$\widehat{\xi}_j^a(k) = \left(\prod_{r=j+1}^{J-1} \nu_k^{r,a}\right) \exp\left(-\frac{k^2+a^2}{(J-2)a}\right).$$

One can rewrite the coefficients $\widehat{\xi}_j^{a,0}(k)$ as

$$\widehat{\xi}_j^{a,0}(k) = \left(\prod_{r=j+1}^{J-1} \nu_k^{r,a,0} \right) \exp\left(-\frac{k^2+a^2}{(J-2)a} \right),$$

where we denote $\nu_k^{r,a,0} := \exp\left(-\frac{k^2+a^2}{(r-1)(r-2)a} \right)$. So,

$$\left| \widehat{\xi}_j^a(k) - \widehat{\xi}_j^{a,0}(k) \right| = \left| \prod_{r=j+1}^{J-1} \nu_k^{r,a} - \prod_{r=j+1}^{J-1} \nu_k^{r,a,0} \right| \exp\left(-\frac{k^2+a^2}{(J-2)a} \right).$$

From $|\nu_k^{r,a,0}| \le 1$, $|\nu_k^{r,a}| \le 1$ it follows that

$$\left| \widehat{\xi}_j^a(k) - \widehat{\xi}_j^{a,0}(k) \right| \le \exp\left(-\frac{k^2+a^2}{(J-2)a} \right).$$

If we combine this estimate with (10), we get

$$\sum_{k\in\mathbb{Z}} k^2 \left| \widehat{\xi}_j^a(k) - \widehat{\xi}_j^{a,0}(k) \right|^2 \le 2 \sum_{n=2^{j-2}+1}^{\infty} n^2 \exp\left(-\frac{2(n^2+a^2)}{(\lfloor \log_2 |n-1| \rfloor + 1)a} \right).$$

The last expression is a remainder of a convergent series. Therefore, it tends to 0 as $j \to \infty$. To check the convergence of the series to 0 as $a \to \infty$ and uniformness on a in the case of $j \to \infty$, we suggest the following estimate

$$\frac{2(x^2+a^2)}{(\log_2(x-1)+1)a} \ge \frac{2(x^2+a^2)}{(\log_2 x + 3)a} \ge a^{1/4} x^{1/2} \quad \text{for all } x \ge 2,\ a \ge 1. \tag{11}$$

To prove (11) rewrite the inequality in the form

$$2a^2 - x^{1/2}(\log_2 x + 3)a^{5/4} + 2x^2 \ge 0$$

and by definition, put $g(a) := 2a^2 - x^{1/2}(\log_2 x + 3)a^{5/4} + 2x^2$. Using standard methods we find $g'(a) = 4a - 5/4x^{1/2}(\log_2 x + 3)a^{1/4}$, and $g'(a) = 0$ as $a = a_0 = (5/16x^{1/2}(\log_2 x + 3))^{4/3}$. Let us check that $\min_{a\ge 1} g(a) = g(a_0) \ge 0$. Indeed, $g(a_0) = 2x^{4/3}\left(x^{2/3} - b(\log_2 x + 3)^{8/3} \right)$, where $b := 3/16(5/16)^{5/3}$. By definition, put

$$g_1(t) = \frac{2^{2t/3}}{b(t+3)^{8/3}}.$$

Then $g_1'(t) = 0$ as $t = t_0 = 4\log_2 \mathrm{e} - 3$ and it is a straightforward calculation to show that $\min_{t\ge 0} g_1(t) = g_1(t_0) \ge 0$. So, $g(a_0) \ge 0$, and (11) is proven.

Using estimate (11), we get

$$\sum_{n=2^{j-2}+1}^{\infty} n^2 \exp\left(-\frac{2(n^2+a^2)}{(\lfloor \log_2 n \rfloor + 3)a} \right) \le \sum_{n=2^{j-2}+1}^{\infty} n^2 \exp\left(-a^{1/4} n^{1/2} \right) \le \sum_{n=1}^{\infty} n^2 \exp\left(-a^{1/4} n^{1/2} \right).$$

It is clear that the last series converges uniformly on $a > 1$. So, it tends to 0 as $a \to \infty$.

Thus, the convergence $\lim_{j\to\infty} \|\xi_j^a - \xi_j^{a,0}\|_{W_1^2} = 0$ is uniformly on $a > 1$ and $\lim_{a\to\infty} \|\xi_j^a - \xi_j^{a,0}\|_{W_1^2} = 0$ is uniformly on $j \in \mathbb{N}$. This completes the proof of (5).

Let us consider the case of the wavelet sequence $(\psi_j^a)_j$. We prove this case in the same manner as the last one. The functions ψ_j^a, η_j^a, $\eta_j^{a,0}$ (the last two functions appear later) play the role of φ_j^a, ξ_j^a and $\xi_j^{a,0}$ respectively.

It follows from the definition of ψ_j^a that $\widehat{\psi}_j^a(k) = \lambda_k^{j+1,a} \widehat{\varphi}_{j+1}^a(k) = \mathrm{e}^{2\pi \mathrm{i} 2^{-j-1}k} \mu_{k+2^j}^{j+1,a} \widehat{\varphi}_{j+1}^a(k) = 2^{1/2} \mathrm{e}^{2\pi \mathrm{i} 2^{-j-1}k} \nu_{k+2^j}^{j+1,a} 2^{(-j-1)/2} \widehat{\xi}_{j+1}^a(k)$. Denote $\widehat{\eta}_j^a(k) := \mathrm{e}^{2\pi \mathrm{i} 2^{-j-1}k} \nu_{k+2^j}^{j+1,a} \widehat{\xi}_{j+1}^a(k)$. Since $\nu_{k+2^j}^{j+1,a} = \sqrt{1 - \exp\left(-2(k^2+a^2)/(j(j-1)a)\right)}$ for $k = -2^{j-1}+1, \dots, 2^{j-1}$, one can introduce a function $\eta_j^{a,0}$ by

$$\widehat{\eta}_j^{a,0}(k) := \mathrm{e}^{2\pi \mathrm{i} 2^{-j-1}k} \sqrt{1 - \exp\left(-\frac{2(k^2+a^2)}{(j(j-1)a)} \right)}\ \widehat{\xi}_{j+1}^{a,0}(k).$$

Then, the functions $\eta_j^{a,0}$ and η_j^a satisfy the property (compare with (10))

$$\widehat{\eta}_j^a(k) = \widehat{\eta}_j^{a,0}(k) \qquad \text{for} \quad k = -2^{j-1}+1, \dots, 2^{j-1}.$$

Using the same arguments as for the scaling sequence, it can be shown that $\lim_{j\to\infty} \|(\eta_j^a)' - (\eta_j^{a,0})'\| = 0$ and $\lim_{a\to\infty} \|(\eta_j^a)' - (\eta_j^{a,0})'\| = 0$ are fulfilled uniformly on $a > 1$ and $j \in \mathbb{N}$ respectively. Therefore, $\lim_{j\to\infty} \sup_{a>1} |UC(\eta_j^a) - UC(\eta_j^{a,0})| = 0$ and $\lim_{a\to\infty} \sup_{j>0} |UC(\eta_j^a) - UC(\eta_j^{a,0})| = 0$. Hence, to conclude the proof it remains to check the equalities $\lim_{j\to\infty} UC(\eta_j^{a,0}) = 3/2$ and $\lim_{a\to\infty} UC(\eta_j^{a,0}) = 3/2$. The proof is given in Lemma 2. This completes also the proof of Theorem 3. $\square$

Lemma 2 *The following equalities hold:*

$$\lim_{j\to\infty} UC(\eta_j^{a,0}) = 3/2 \qquad \textit{for any fixed } a>1$$

and

$$\lim_{a\to\infty} UC(\eta_j^{a,0}) = 1/2 \qquad \textit{for any fixed } j\in\mathbb{N}.$$

Proof. Let us estimate the quantities $((\eta_j^{a,0})', \eta_j^{a,0})$, $\|\eta_j^{a,0}\|^2$, $\|(\eta_j^{a,0})'\|^2$, and $|\tau(\eta_j^{a,0})|$ as $j\to\infty$ for a fixed $a>1$ and as $a\to\infty$ for any fixed $j\in\mathbb{N}$. Since $|\widehat{\eta}_j^{a,0}(k)| = |\widehat{\eta}_j^{a,0}(-k)|$, we see that

$$((\eta_j^{a,0})', \eta_j^{a,0}) = \sum_{k\in\mathbb{Z}} k\ |\widehat{\eta}_j^{a,0}(k)|^2 = 0.$$

For convenience we replace j by $1/h$ and a by $1/q$ in our calculations. Then for the new parameters we have $h\to 0$ and $q\to 0$. However, to avoid the fussiness of notations we keep the former name for the function $\eta_j^{a,0}$. We obtain

$$\begin{aligned}\|\eta_j^{a,0}\|^2 = \sum_{k\in\mathbb{Z}} |\widehat{\eta}_j^{a,0}(k)|^2 &= \sum_{k\in\mathbb{Z}} \left(1-\exp\left(-\frac{2h^2(q^2k^2+1)}{(1-h)q}\right)\right)\exp\left(-\frac{2h(q^2k^2+1)}{q}\right)\\ &= \sum_{k\in\mathbb{Z}} \left(\exp\left(-\frac{2h(q^2k^2+1)}{q}\right) - \exp\left(-\frac{2h(q^2k^2+1)}{(1-h)q}\right)\right).\end{aligned}$$

Using the Poisson summation formula for the function $f(t) = \mathrm{e}^{-bt^2}$

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{-b(k-t)^2} = \sqrt{\frac{\pi}{b}} \sum_{k\in\mathbb{Z}} \cos 2\pi kt\ \mathrm{e}^{\frac{-\pi^2k^2}{b}}, \tag{12}$$

we get with $b=2hq$, $t=0$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{2h(q^2k^2+1)}{q}\right) &= \mathrm{e}^{-\frac{2h}{q}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-2hqk^2} = \mathrm{e}^{-\frac{2h}{q}} \sqrt{\frac{\pi}{2hq}} \sum_{k\in\mathbb{Z}} \exp\left(\frac{-\pi^2k^2}{2hq}\right)\\ &= \mathrm{e}^{-\frac{2h}{q}} \sqrt{\frac{\pi}{2hq}} \left(1+2\mathrm{e}^{\frac{-\pi^2}{2hq}} \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2(k^2-1)}{2hq}\right)\right) = \mathrm{e}^{-\frac{2h}{q}} \sqrt{\frac{\pi}{2hq}} + \mathrm{e}^{\frac{-1}{hq}} O(hq).\end{aligned} \tag{13}$$

The last equality follows from the fact that the series $\sum_{k=1}^{\infty} \exp\left(-\pi^2(k^2-1)/(2hq)\right)$ is bounded by the geometric series $\sum_{k=1}^{\infty} \exp\left(-\pi^2(k-1)/(2hq)\right)$. We replace $\mathrm{e}^{-\pi^2/(2hq)}$ by $\mathrm{e}^{-1/(hq)}$ to unify the estimates for all expressions.

Using formula (12) for $b = 2hq/(1-h)$ and $t=0$, we obtain in the same way

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \exp\left(-\frac{2h(q^2k^2+1)}{(1-h)q}\right) &= \exp\left(-\frac{2h}{q(1-h)}\right) \sqrt{\frac{\pi(1-h)}{2hq}} \sum_{k\in\mathbb{Z}} \exp\left(\frac{-\pi^2k^2(1-h)}{2hq}\right)\\ &= \exp\left(-\frac{2h}{q(1-h)}\right) \sqrt{\frac{\pi(1-h)}{2hq}} \left(1+2\exp\left(\frac{-\pi^2(1-h)}{2hq}\right) \sum_{k=1}^{\infty} \exp\left(\frac{-\pi^2(1-h)(k^2-1)}{2hq}\right)\right)\\ &= \exp\left(-\frac{2h}{q(1-h)}\right) \left(\sqrt{\frac{\pi(1-h)}{2hq}} + \mathrm{e}^{\frac{-1}{hq}} O(hq)\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\eta_j^{a,0}\|^2 &= \sum_{k\in\mathbb{Z}} \left(\exp\left(-\frac{2h(q^2k^2+1)}{q}\right) - \exp\left(-\frac{2h(q^2k^2+1)}{(1-h)q}\right)\right)\\ &= \mathrm{e}^{-\frac{2h}{q}} \sqrt{\frac{\pi}{2hq}} + \mathrm{e}^{-\frac{2h}{q(1-h)}} \sqrt{\frac{\pi(1-h)}{2hq}} + \mathrm{e}^{\frac{-1}{hq}} O(hq).\end{aligned} \tag{14}$$

To estimate the quantities $\|(\eta_j^{a,0})'\|^2$ we write

$$\begin{aligned}\frac{1}{4\pi^2}\|(\eta_j^{a,0})'\|^2 &= \sum_{k\in\mathbb{Z}} k^2 |\widehat{\eta}_j^{a,0}(k)|^2 = \sum_{k\in\mathbb{Z}} k^2 \left(\exp\left(-\frac{2h(q^2k^2+1)}{q}\right) - \exp\left(-\frac{2h(q^2k^2+1)}{(1-h)q}\right)\right)\\ &= \mathrm{e}^{-\frac{2h}{q}} \sum_{k\in\mathbb{Z}} k^2 \mathrm{e}^{-2hqk^2} - \mathrm{e}^{-\frac{2h}{q(1-h)}} \sum_{k\in\mathbb{Z}} k^2 \exp\left(-\frac{2hqk^2}{1-h}\right).\end{aligned}$$

Using formula (12) again with $b=2hq$, $t=0$, we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}}k^2\mathrm{e}^{-2hqk^2} &= -\frac{1}{2q}\left(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-2hqk^2}\right)'_h=-\frac{1}{2q}\left(\sqrt{\frac{\pi}{2hq}}\sum_{k\in\mathbb{Z}}\exp\left(\frac{-\pi^2k^2}{2hq}\right)\right)'_h\\ &= -\frac{1}{2q}\left(-\frac{1}{2}\right)\sqrt{\frac{\pi}{2hq}}\frac{1}{h}\sum_{k\in\mathbb{Z}}\exp\left(\frac{-\pi^2k^2}{2hq}\right)-\frac{1}{2q}\sqrt{\frac{\pi}{2hq}}\frac{\pi^2}{2h^2q}\sum_{k\in\mathbb{Z}}k^2\exp\left(\frac{-\pi^2k^2}{2hq}\right).\end{aligned}$$

Note that it is possible to change the order of summation and differentiation above. If we combine the equality with (13), we get

$$\sum_{k\in\mathbb{Z}}k^2\mathrm{e}^{-2hqk^2}=\frac{1}{4q}\sqrt{\frac{\pi}{2hq}}\frac{1}{h}\left(1+\mathrm{e}^{-\frac{1}{hq}}O(hq)\right)-\frac{1}{2q}\sqrt{\frac{\pi}{2hq}}\frac{\pi^2}{h^2q}\mathrm{e}^{\frac{-\pi^2}{2hq}}\sum_{k=1}^{\infty}k^2\exp\left(\frac{-\pi^2(k^2-1)}{2hq}\right).$$

Since the quantity $k^2\exp\left(-\pi^2(k^2-1)/(4hq)\right)$ is bounded and the series $\sum_{k=1}^{\infty}\exp\left(-\pi^2(k^2-1)/(4hq)\right)$ is majorized by $\sum_{k=1}^{\infty}\exp\left(-\pi^2(k-1)/(4hq)\right)$, we conclude that the series $\sum_{k=1}^{\infty}k^2\exp\left(-\pi^2(k^2-1)/(2hq)\right)$ is bounded. Hence,

$$\frac{1}{2q}\sqrt{\frac{\pi}{2hq}}\frac{\pi^2}{h^2q}\exp\left(\frac{-\pi^2}{2hq}\right)\sum_{k=1}^{\infty}k^2\exp\left(\frac{-\pi^2(k^2-1)}{2hq}\right)=\mathrm{e}^{-\frac{1}{hq}}O(hq).$$

Therefore,

$$\sum_{k\in\mathbb{Z}}k^2\mathrm{e}^{-2hqk^2}=\frac{1}{4}\sqrt{\frac{\pi}{2hq}}\frac{1}{hq}+\mathrm{e}^{-\frac{1}{hq}}O(hq).$$

If in the last calculation we replace $2h$ by $2h/(1-h)$, we obtain

$$\sum_{k\in\mathbb{Z}}k^2\exp\left(-\frac{2hqk^2}{1-h}\right)=-\frac{(1-h)^2}{2q}\left(\sum_{k\in\mathbb{Z}}\exp\left(-\frac{2hqk^2}{1-h}\right)\right)'_h$$

$$=-\frac{(1-h)^2}{2q}\left(\sqrt{\frac{\pi(1-h)}{2hq}}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{\pi^2k^2(1-h)}{2hq}\right)\right)'_h=\frac{1}{4}\sqrt{\frac{\pi}{2hq}}\frac{1}{hq}(1-h)\sqrt{1-h}+\mathrm{e}^{-\frac{1}{hq}}O(hq).$$

Hence,

$$\begin{aligned}\frac{1}{4\pi^2}\|(\eta_j^{a,0})'\|^2 &= \mathrm{e}^{-\frac{2h}{q}}\sum_{k\in\mathbb{Z}}k^2\mathrm{e}^{-2hqk^2}-\mathrm{e}^{-\frac{2h}{q(1-h)}}\sum_{k\in\mathbb{Z}}k^2\exp\left(-\frac{2hqk^2}{1-h}\right)\\ &= \mathrm{e}^{-\frac{2h}{q}}\frac{1}{4qh}\sqrt{\frac{\pi}{2hq}}-\mathrm{e}^{-\frac{2h}{q(1-h)}}\frac{1}{4hq}\sqrt{\frac{\pi}{2hq}}(1-h)^{3/2}+\mathrm{e}^{-\frac{1}{hq}}O(hq).\end{aligned}$$

So, recalling that $((\eta_j^{a,0})',\,\eta_j^{a,0})=0$, we get the following expression for the frequency variance:

$$\frac{1}{4\pi^2}\frac{\|(\eta_j^{a,0})'\|^2}{\|\eta_j^{a,0}\|^2}=\frac{1}{4hq}\frac{\mathrm{e}^{-\frac{2h}{q}}-\mathrm{e}^{-\frac{2h}{q(1-h)}}(1-h)^{3/2}}{\mathrm{e}^{-\frac{2h}{q}}-\mathrm{e}^{-\frac{2h}{q(1-h)}}(1-h)^{1/2}}.$$

Therefore,

$$\frac{1}{4\pi^2}\frac{\|(\eta_j^{a,0})'\|^2}{\|\eta_j^{a,0}\|^2}\sim\frac{3}{4hq}\quad\text{as } h\to 0\qquad\text{and}\qquad\frac{1}{4\pi^2}\frac{\|(\eta_j^{a,0})'\|^2}{\|\eta_j^{a,0}\|^2}\sim\frac{1}{4hq}\quad\text{as } q\to 0. \tag{15}$$

Let us, finally, estimate the first trigonometric moment $\tau(\eta_j^{a,0})$. By the definition of $\tau(\eta_j^{a,0})$ we have

$$\begin{aligned}\frac{1}{2\pi}|\tau(\eta_j^{a,0})| &= \Bigg|\exp\left(-2\pi\mathrm{i}2^{-\frac{1+h}{h}}\right)\sum_{k\in\mathbb{Z}}\sqrt{\left(1-\exp\left(-\frac{2(k^2q^2+1)h^2}{(1-h)q}\right)\right)\left(1-\exp\left(-\frac{2(q^2(k+1)^2+1)h^2}{(1-h)q}\right)\right)}\\ &\quad\times\exp\left(-\frac{h(q^2k^2+1)}{q}\right)\exp\left(-\frac{h(q^2(k+1)^2+1)}{q}\right)\Bigg|\\ &= \mathrm{e}^{-\frac{2h}{q}}\sum_{k\in\mathbb{Z}}\sqrt{\left(1-\exp\left(-\frac{2h^2}{(1-h)q}\right)\exp\left(-\frac{2k^2qh^2}{1-h}\right)\right)\left(1-\exp\left(-\frac{2h^2}{(1-h)q}\right)\exp\left(-\frac{2(k+1)^2qh^2}{1-h}\right)\right)}\\ &\quad\times\mathrm{e}^{-hq(2k^2+2k+1)}\end{aligned} \tag{16}$$

Our task is to get the following representation for $|\tau(\eta_j^{a,0})|$:

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})| = \mathrm{e}^{-\frac{2h}{q}}\left(\sqrt{\frac{\pi}{8q}}\sqrt{h}+\frac{9q\sqrt{\pi}}{4\sqrt{2}}\sqrt{h^3}+O(h^2\ln h)\right)\quad\text{for a fixed } q<1 \text{ and } h\to 0; \tag{17}$$

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})| = \mathrm{e}^{-\frac{2h}{q}}\left(\mathrm{e}^{-\frac{hq}{2}}\sqrt{\frac{\pi}{2hq}}+O\Big(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}}\Big)\right)\quad\text{for a fixed } h<1 \text{ and } q\to 0. \tag{18}$$

Let us prove the estimate (17). Put by definition

$$a := \frac{2h^2}{1-h},\ v(k) := qk^2 + \frac{1}{q},\ s(k) := 2k^2 + 2k + 1.$$

Thus, the first trigonometric moment is rewritten as follows:

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})| = \mathrm{e}^{-\frac{2h}{q}} \sum_{k\in\mathbb{Z}} \sqrt{(1-\mathrm{e}^{av(k)})(1-\mathrm{e}^{av(k+1)})}\mathrm{e}^{-hqs(k)}.$$

Using the Taylor formula for the function $f(a) = \sqrt{(1-\mathrm{e}^{av(k)})(1-\mathrm{e}^{av(k+1)})}$ in the neighborhood of $a = 0$, we get

$$f(a) = \sqrt{v(k)v(k+1)}a - \frac{1}{4}\sqrt{v(k)v(k+1)}(v(k)+v(k+1))a^2 + \frac{f'''(\bar{a})}{6}a^3,$$

where

$$\begin{aligned} f'''(a) \quad = \quad & \frac{1}{8(1-\mathrm{e}^{-av(k)})^{5/2}(1-\mathrm{e}^{-av(k+1)})^{5/2}} \Big(\mathrm{e}^{-av(k)}(1-\mathrm{e}^{-av(k)})^2(1-\mathrm{e}^{-av(k+1)})^3 v^3(k) \\ & +3\mathrm{e}^{-av(k)}(1-\mathrm{e}^{-av(k+1)})^2 v^2(k)\Big((1-\mathrm{e}^{-av(k+1)})v(k) - \mathrm{e}^{-av(k+1)}(1-\mathrm{e}^{-av(k)})v(k+1)\Big) \\ & -3\mathrm{e}^{-av(k)-av(k+1)}(1-\mathrm{e}^{-av(k)})^2(1-\mathrm{e}^{-av(k+1)})^2 v(k)v(k+1)(v(k)+v(k+1)) \\ & +3\mathrm{e}^{-av(k+1)}(1-\mathrm{e}^{-av(k)})^2 v^2(k+1)\Big((1-\mathrm{e}^{-av(k)})v(k+1) - \mathrm{e}^{-av(k)}(1-\mathrm{e}^{-av(k+1)})v(k)\Big) \\ & +\mathrm{e}^{-av(k+1)}(1-\mathrm{e}^{-av(k+1)})^2(1-\mathrm{e}^{-av(k)})^3 v^3(k+1)\Big). \end{aligned}$$

It is a straightforward calculation to show the following estimate:

$$|f'''(a)| \le C_1 v^{5/2}(k)v^{1/2}(k+1) + C_2 v^{3/2}(k)v^{3/2}(k+1) + C_3 v^{1/2}(k)v^{5/2}(k+1).$$

In the sequel, we denote by C_n, $n \in \mathbb{N}$ positive constants. Recalling the notations for a, $v(k)$ we obtain

$$\frac{|f'''(\bar{a})|}{6}a^3 = O(k^6h^6).$$

Using the ideas of estimating $\|(\eta_j^{a,0})'\|^2$, we have

$$h^6\sum_{k\in\mathbb{Z}} k^6\mathrm{e}^{-hqs(k)} \le C_4 h^6 \sum_{k\in\mathbb{Z}} k^6 \mathrm{e}^{-hqk^2} = C_5 h^6 \left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-hqk^2}\right)'''_{h^3} = C_6 h^6\left(h^{-7/2} + \mathrm{e}^{-\frac{1}{hq}}O(hq)\right) = O(h^{5/2}).$$

Therefore, the first trigonometric moment takes the form

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})| = \mathrm{e}^{-\frac{2h}{q}}\left|\sum_{k\in\mathbb{Z}} \sqrt{v(k)v(k+1)}\left(a - \frac{1}{4}(v(k)+v(k+1))a^2\right)\mathrm{e}^{-hqs(k)}\right| + O(h^{5/2}).$$

Let $\alpha := 1/k$ and let the function g defined by

$$\sqrt{v(k)v(k+1)} = \sqrt{\left(qk^2 + \frac{1}{q}\right)\left(q(k+1)^2 + \frac{1}{q}\right)} = k^2\sqrt{\left(q + \frac{\alpha^2}{q}\right)\left(q(\alpha+1)^2 + \frac{\alpha^2}{q}\right)} =: k^2\ g(\alpha).$$

Using the Taylor formula for $g(\alpha)$ in the neighborhood of $\alpha = 0$, we obtain

$$g(\alpha) = q + q\alpha + \frac{1}{q}\alpha^2 + \frac{g'''(\bar{\alpha})}{6}\alpha^3,$$

where

$$\begin{aligned} g'''(\alpha) \quad = \quad & \frac{1}{2g(\alpha)}\frac{6\alpha(2/q+2q) + 6(2\alpha/q + 2(1+\alpha)q)}{q} \\ & -\frac{3}{4g^3(\alpha)}\left((\alpha^2/q+q)(2/q+2q) + \frac{4\alpha(2\alpha/q + 2(1+\alpha)q) + 4(\alpha^2/q + (1+\alpha)^2 q)}{q}\right) \\ & \times\left((\alpha^2/q+q)(2\alpha/q + 2(1+\alpha)q) + \frac{2\alpha(\alpha^2/q + (1+\alpha)^2q)}{q}\right) \\ & +\frac{3}{8g^5(\alpha)}\left((\alpha^2/q+q)(2\alpha/q + 2(1+\alpha)q) + \frac{2\alpha(\alpha^2/q + (1+\alpha)^2q)}{q}\right)^3. \end{aligned}$$

It is easy to see that $\lim_{\alpha\to 0} g(\alpha)=0$, $\lim_{\alpha\to\infty} g(\alpha)=0$, and $g(\alpha)$ is continuous for $\alpha>0$ for a fixed $0<q<1$. Thus, $|g'''(\bar\alpha)|<C(q)$ and $\frac{g'''(\bar\alpha)}{6}\alpha^3=O(1)\alpha^3$. Therefore, for the first trigonometric moment we have

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})|=2\mathrm{e}^{-\frac{2h}{q}}\left|\sum_{k=1}^{\infty}\left(k^2q+kq+\frac{1}{q}+\frac{O(1)}{k}\right)\left(a-\frac{1}{4}(v(k)+v(k+1))a^2\right)\mathrm{e}^{-hqs(k)}\right|+O(h^2).$$

In the latter expression, we omitted the summand for $k=0$ which can be estimated as

$$\mathrm{e}^{-\frac{2h}{q}}\left(\sqrt{1/q(q+1/q)}a-1/4\sqrt{1/q(q+1/q)}a^2\right)\mathrm{e}^{-hq}=O(h^2).$$

Let us estimate the coefficient of $O(1)$ in the first trigonometric moment using $a=2h^2/(1-h)$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k}\left(a-\frac{1}{4}\left(qs(k)+\frac{2}{q}\right)a^2\right)\mathrm{e}^{-hqs(k)} &\sim C_1a\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{e}^{-hqk^2}=C_1a\left(\mathrm{e}^{-hq}+\sum_{k=2}^{\infty}\frac{1}{k}\mathrm{e}^{-hqk^2}\right)\\ &\le C_1a\left(\mathrm{e}^{-hq}+\int_1^{\infty}\frac{1}{x}\mathrm{e}^{-hqx^2}\,\mathrm{d}x\right)=C_1a\left(\mathrm{e}^{-hq}+\int_{\sqrt{hq}}^{\infty}\frac{1}{x}\mathrm{e}^{-x^2}\,\mathrm{d}x\right)\\ &= C_1a\left(\mathrm{e}^{-hq}-\mathrm{e}^{-h^2q^2}\ln(hq)+\int_{\sqrt{hq}}^{\infty}2x\mathrm{e}^{-x^2}\ln x\,\mathrm{d}x\right)=O(h^2\ln h).\end{aligned}$$

Finally, recalling the notation $s(k)=2k^2+2k+1$, we get the following expression for the first trigonometric moment

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})|=2\mathrm{e}^{-\frac{2h}{q}}\left|\sum_{k\in\mathbb{Z}}\left(\frac{2h^2}{1-h}\right)^2\left(\frac{q}{2}s(k)+\frac{1}{q}-\frac{q}{2}\right)\left(1-\frac{1}{4}\left(qs(k)+\frac{2}{q}\right)\frac{2h^2}{1-h}\right)\mathrm{e}^{-hqs(k)}\right|+O(h^2\ln h). \tag{19}$$

Using the Poisson summation formula and differentiating the series $\sum_{k\in\mathbb{Z}}\mathrm{e}^{-hqs(k)}$ appropriate times as it is done to estimate $\|(\eta_j^{a,0})'\|^2$ we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\mathrm{e}^{-hqs(k)} &= \mathrm{e}^{-\frac{hq}{2}}\sqrt{\frac{\pi}{2hq}}+\mathrm{e}^{-\frac{1}{hq}}O(hq),\\ \sum_{k\in\mathbb{Z}}s(k)\mathrm{e}^{-hqs(k)} &= \frac{1+hq}{2hq}\mathrm{e}^{-\frac{hq}{2}}\sqrt{\frac{\pi}{2hq}}+\mathrm{e}^{-\frac{1}{hq}}O(hq),\\ \sum_{k\in\mathbb{Z}}(s(k))^2\mathrm{e}^{-hqs(k)} &= \frac{h^2q^2+2hq+3}{4h^2q^2}\mathrm{e}^{-\frac{hq}{2}}\sqrt{\frac{\pi}{2hq}}+\mathrm{e}^{-\frac{1}{hq}}O(hq).\end{aligned}$$

Substituting these expressions into (19) we finally obtain (17).

Let us prove (18). We estimate $|\tau(\eta_j^{a,0})|$ for a fixed $0<h<1$ and $q\to 0$. We start with expression (16). Put by definition

$$x:=\exp\left(-\frac{2h^2(q^2k^2+1)}{(1-h)q}\right),\qquad y:=\exp\left(-\frac{2h^2(q^2(k+1)^2+1)}{(1-h)q}\right).$$

It is clear that $0<x,\ y<1/2$, and $\lim_{q\to 0}x=\lim_{q\to 0}y=0$. So,

$$\sqrt{(1-x)(1-y)}\le(1-C_1x)(1-C_2y).$$

Therefore, $\frac{1}{2\pi}|\tau(\eta_j^{a,0})|$ is rewritten as follows:

$$\begin{aligned}\frac{1}{2\pi}|\tau(\eta_j^{a,0})| &= \mathrm{e}^{-\frac{2h}{q}}\sum_{k\in\mathbb{Z}}(1-C_1x)(1-C_2y)\mathrm{e}^{-hqs(k)}\\ &= \mathrm{e}^{-\frac{2h}{q}}\left(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-hqs(k)}-C_1\sum_{k\in\mathbb{Z}}x\mathrm{e}^{-hqs(k)}-C_2\sum_{k\in\mathbb{Z}}y\mathrm{e}^{-hqs(k)}+C_1C_2\sum_{k\in\mathbb{Z}}xy\mathrm{e}^{-hqs(k)}\right)=\mathrm{e}^{-\frac{2h}{q}}(S_1+S_2+S_3+S_4).\end{aligned}$$

It turns out that $S_n=O(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}})$ for $n=2,3,4$. Indeed, using the Poisson summation formula as it is done to estimate $\|\eta_j^{a,0}\|^2$ we get

$$S_2=C_1\sum_{k\in\mathbb{Z}}x\mathrm{e}^{-hqs(k)}=C_1\mathrm{e}^{-\frac{2h^2}{(1-h)q}}\sum_{k\in\mathbb{Z}}\exp\left(-\left(-\frac{2h^2q}{1-h}-2hq\right)k^2-2hqk-hq\right)=O\left(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}}\right).$$

Therefore,

$$\frac{1}{2\pi}|\tau(\eta_j^{a,0})|=\mathrm{e}^{-\frac{2h}{q}}\left(S_1+O\left(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}}\right)\right)=\mathrm{e}^{-\frac{2h}{q}}\left(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-hqs(k)}+O\left(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}}\right)\right)=\mathrm{e}^{-\frac{2h}{q}}\left(\mathrm{e}^{-\frac{hq}{2}}\sqrt{\frac{\pi}{2hq}}+O\left(\frac{1}{\sqrt{q}}\mathrm{e}^{-\frac{1}{q}}\right)\right)$$

and (18) is proven.

Finally, substituting expressions (14), (15), (17), and (18) into the definition of the periodic uncertainty constant (see Definition 1) and calculating the limits we obtain

$$\lim_{j\to\infty} UC(\psi_{j,a}) = \frac{3}{2}, \qquad \lim_{a\to\infty} UC(\psi_{j,a}) = \frac{1}{2}.$$

This completes the proof of Lemma 2. $\square$

4 Discussion

In Theorem 3 for the case of $j \to \infty$ we get an optimal uncertainty constant 1/2 for the constructed scaling sequences, but it turns out that the wavelet sequences have uncertainty constants equal to 3/2 only. This gives rise to a discussion. Let $\psi^0 \in L_2(\mathbb{R})$ be a wavelet on the real line. Put by definition

$$\psi^p_{j,k}(x) := 2^{j/2} \sum_{n\in\mathbb{Z}} \psi^0(2^j(x+n)+k).$$

The sequence $\psi^p_{j,k}$ is said to be a periodic wavelet set generated by periodization. It can easily be deduced that under a mild restriction the Breitenberger uncertainty constant is greater or equal to 3/2 for any periodic wavelet generated by periodization. Namely we get the following

Theorem 4 *Suppose* $\{2^{j/2}\psi^0(2^j\cdot -k):\ j,k\in\mathbb{Z}\}$ *forms a Bessel sequence and* $\omega_{0,\widehat{\psi^0}} = ((\psi^0)',\, \psi^0)_{L_2(\mathbb{R})} = 0$, *then*

$$\lim_{j\to\infty} UC(\psi^p_{j,k}) \geq 3/2.$$

Proof. In [2] the following result is proven: for any function ψ generating an orthogonal wavelet basis defined on $\mathbb{R}$ such that $(\psi',\, \psi)_{L_2(\mathbb{R})} = 0$, the Heisenberg uncertainty constant is greater or equal to 3/2. In [1] the last result is extended to any function generating a wavelet Bessel set. So, we get the inequality $UC_H(\psi^0) \geq 3/2$. On the other hand, it is proved in [18] that for periodic wavelets generated by periodization of a wavelet function on the real line the periodic uncertainty constant tends to the real line uncertainty constant of the original function as a parameter of periodization tends to infinity. Thus, $\lim_{j\to\infty} UC(\psi^p_{j,k}) = UC_H(\psi^0)$. This concludes the proof of Theorem 4. $\square$

These arguments motivate a conjecture: if $(\psi'_j,\, \psi_j)_{L_2(0,\,1)} = 0$, then $\lim_{j\to\infty} UC(\psi_j) \geq 3/2$ for any periodic wavelet sequence $(\psi_j)_j$. If this is true, the family of normalized tight wavelet frames constructed in Theorem 3 has the optimal uncertainty constant. To prove the conjecture is a task for a future investigation.

One can prove some results in the direction of Theorem 3 starting with a wavelet on the real line and using the periodization (as it is described in [18]). However, it is unknown if there exists a real-line orthonormal wavelet basis possessing the Heisenberg uncertainty constant UC_H less than 2.134. This value is attained for a Daubechies wavelet [7]. The smallest possible value of UC_H for the family of the Meyer wavelets equals to 6.874 [13]. It is well known [5] that the uncertainty constant UC_H of Battle-Lemarie and Daubechies wavelets tends to infinity as their orders grow.

On the other hand, there are some examples of real-line wavelet frames possessing asymptotically optimal uncertainty constants. In [22] it is proven that nonorthogonal B-spline wavelets tend to Gabor functions having optimal uncertainty constants. In [11] some conditions on wavelet masks are established to provide a minimal possible value for the uncertainty constant. They are, however, too restrictive, and it is unknown if there exists a nontrivial example. Anyway, suppose $f_n \in L_2(\mathbb{R})$, $n \in \mathbb{N}$ is a sequence such that $\lim_{n\to\infty} UC_H(f_n) = 1/2$. Using the periodization we define sequences of periodic functions $f^p_{j,n}(x) := \sum_{k\in\mathbb{Z}} f_n(2^j(x+k))$ and applying results from [18] we get only

$$\lim_{j\to\infty} \lim_{n\to\infty} UC(f^p_{j,n}) = 1/2.$$

However, it is weaker than the equalities of the form (5), (6).

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References

[1] R. Balan, An uncertainty inequality for wavelet sets, Appl. Comput. Harmon. Anal. 5 (1998) 106–108.

[2] G. Battle, Heisenberg inequalities for wavelet states, Appl. Comput. Harmon. Anal. 4 (1997) 119–146.

[3] E. Breitenberger, Uncertainty measures and uncertainty relations for angle observables, Found. Phys. 15 (1985) 353–364.

[4] O. Christensen, An Introduction to Frames and Riesz Bases, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.

[5] C.K. Chui, J. Wang, High-order orthonormal scaling functions and wavelets give poor time-frequency localization, J. Fourier Anal. Appl. 2 (1996) 415–426.

[6] I. Daubechies, Ten Lectures on Wavelets, CBMS Series, SIAM, 1992.

[7] Ž. Gimbutas, A. Bastys, Daubechies compactly supported wavelets with minimal Heisenberg boxes, Lit. Math. J. 35 (1995) 343–362.

[8] S.S. Goh, B. Han, Z. Shen, Tight periodic wavelet frames and approximation orders, Appl. Comput. Harmon. Anal. 31 (2010) 228–248.

[9] S.S. Goh, K.M. Teo, Extension principles for tight wavelet frames of periodic functions, Appl. Comput. Harmon. Anal. 25 (2008) 168–186.

[10] S.S. Goh, Ch.H. Yeo, Uncertainty products of local periodic wavelets, Adv. Comput. Math. 13 (2000) 319–333.

[11] T.N.T. Goodman, S.L. Lee, Asymptotic optimality in time-frequency localization of scaling functions and wavelets, in: Frontiers in Interpolation and Application, N.K. Govil, H.N. Mhaskar, R.N. Mohapatra, Z. Nashed, and J. Szadados (Eds.), (2007) 145–171.

[12] Y.W. Koh, S.L. Lee, H.H. Tan, Periodic orthogonal splines and wavelets, Appl. Comput. Harmon. Anal. 2 (1995) 201–218.

[13] E.A. Lebedeva, V.Yu. Protasov, Meyer wavelets with least uncertainty constant, Math. Notes 84 (2008) 680–687.

[14] F.J. Narcowich, J.D. Ward, Wavelets associated with periodic basis functions, Appl. Comput. Harmon. Anal. 3 (1996) 40–56.

[15] I.Ya. Novikov, V.Yu. Protasov, M.A. Skopina, Wavelet Theory, Translations of Mathematical Monographs 239, AMS, 2011.

[16] G. Plonka, M. Tasche, A unified approach to periodic wavelets, in Wavelets: Theory, Algorithms, and Applications, C. Chui, L. Montefusco, and L. Puccio (Eds.), Academic Press, New York (1994) 137–151.

[17] J. Prestin, E. Quak, Optimal functions for a periodic uncertainty principle and multiresolution analysis, Proc. Edinb. Math. Soc., II. Ser. 42 (1999) 225–242.

[18] J. Prestin, E. Quak, H. Rauhut, K. Selig, On the connection of uncertainty principles for functions on the circle and on the real line, J. Fourier Anal. Appl. 9 (2003) 387–409.

[19] H. Rauhut, Best time localized trigonometric polynomials and wavelets, Adv. Comput. Math. 22 (2005) 1–20.

[20] K. Selig, Trigonometric wavelets and the uncertainty principle, in: Approximation Theory, M. W. Müller, M. Felten, and D. H. Mache (Eds.), Math. Research, Vol. 86, Akademie Verlag, Berlin (1995) 293–304.

[21] M. Skopina, Multiresolution analysis of periodic functions, East J. Approx. 3 (1997) 203–224.

[22] M. Unser, A. Aldroubi, M. Eden, On the asymptotic convergence of B-spline wavelets to Gabor functions, IEEE Trans. Inf. Theory 38 (1992) 864–872.